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Comments on 'Bose–Einstein condensation in an Einstein universe'

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Abstract. Using a Mellin integral transform we obtain the exact bulk and finite size terms of the logarithm of the grand partition function of a spinless relativistic boson gas confined in an Einstein universe. The finite size contribution calculated by Al'taie is shown to be negligible when compared with the leading relativistic correction to bulk term unless $N(kT/m) \leqslant 1$. We also verified that the transition temperature is lowered by the relativistic corrections overwhelming the shift produced by the surface effects.

In the ultra-relativistic limit the finite size contribution is shown to be comparable only to higher-order mass corrections to the bulk term.

1. Introduction

The behaviour of an ideal boson gas confined to the background geometry of an Einstein universe has been recently investigated by Al'taie (1978). The system is treated from the very beginning in the non-relativistic approximation and the finite size corrections to the bulk specific heat and number of condensed particles are obtained using the formalism developed by Pathria (1972).

Apart from the expected effect of smoothing out the singularities of the thermodynamic functions at the transition temperature he also verified a displacement of the specific heat maximum towards higher temperatures and an enhancement of the number of condensed particles due to the finite size of the system. In this paper we present a more complete analysis which includes relativistic effects for the spinless gas. The calculations for the case of particles with spin can be carried out by the same procedure. The non-relativistic and ultra-relativistic limits are obtained from the exact expression for the logarithm of the grand partition fuı ction which contains both the bulk and finite size contributions. The finite size term is shown to be very small when compared with the relativistic corrections to the leading bulk terms and should play a very modest role, except in very special cases, in the thermodynamic behaviour of the system.

2. The spinless gas

Using the Mellin Transform we write the logarithm of the grand partition function Ξ of an ideal boson gas as an integral transform of the single particle partition function $Z_1(\beta)$

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as follows (Goulart Rosa and Grandy 1973)

$$\ln \Xi = \frac{-1}{2\mathrm{i}} \int_{\alpha - \mathrm{i}\infty}^{\alpha + \mathrm{i}\infty} t^{-1} \cot (\pi t) \exp (\beta \xi t) \sum_n \exp (-\beta t E_n) \, \mathrm{d}t \tag{1}$$

where $\beta = kT$ (k is the Boltzmann constant, T is the absolute temperature), ξ is the chemical potential, E_n is the energy of the nth single-particle state and α is an arbitrary positive number less than one. The spinless boson of mass 'm' confined to the background of an Einstein universe is described by the scalar field Φ satisfying the equation

$$\partial_\mu \, \partial^\mu \Phi + \left(\frac{1}{a^2} + m^2\right)\Phi = 0 \qquad \mu = 0, 1, 2, 3 \tag{2}$$

which has eigenvalues given by

$$E_n = \frac{1}{a}[n^2 + (ma)^2]^{1/2} \tag{3}$$

with degeneracy $g_n = n^2$, $n = 1, 2, 3, \ldots$, and 'a' is the radius of (spherical) spatial part of the Einstein universe. The sum $Z_1(\beta) = \Sigma_n \exp(-\beta E_n)$ in equation (1) is evaluated using the Poisson summation formula which reads in this case as follows:

$$Z_1(\beta) = Z_{\mathrm{b}}(\beta) + Z_{\mathrm{s}}(\beta) = \int_0^\infty t^2 \exp[-(\beta/a)(t^2 + m^2 a^2)^{1/2}] \, \mathrm{d}t$$

$$+ 2 \sum_{p=1}^{\infty} \int_0^\infty t^2 \exp[-(\beta/a)(t^2 + m^2 a^2)^{1/2}] \cos(2\pi p) \, \mathrm{d}t. \tag{4}$$

The first integral represents the bulk contribution and the Fourier cosine series is the finite size term of the single-particle partition function.

The integrals are evaluated in terms of the modified Bessel functions $K_\nu(t)$ (Gradshteyn and Ryzhik 1965)

$$Z_{\mathrm{b}}(\beta) = (ma)^3 K_2(\beta m)/\beta m \tag{5}$$

$$Z_{\mathrm{s}}(\beta) = 2 \sum_{p=1}^{\infty} \frac{m^4 a^3 \beta}{\tau^2}\left(K_2(\tau) - \frac{(2\pi pma)^2}{\tau} K_3(\tau)\right) \tag{6}$$

where

$$\tau = ma[(2\pi p)^2 + (\beta/a)^2]^{1/2}.$$

We recall that the modified Bessel functions are regular functions throughout the t plane cut along the negative axis and are real and positive when $\nu > -1$ and $t > 0$. The asymptotic expansion for large argument and the ascending series from which one can obtain the behaviour in the limit of $t \to 0$ are, respectively,

$$K_\nu(t) \sim (\pi/2t)^{1/2} \exp(-t)\left(1 + \frac{4\nu^2 - 1}{8t} + \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{2!8t^2} + \cdots\right) \tag{7}$$

$$K_\nu(t) = \tfrac{1}{2}(t/2)^{-\nu} \sum_{k=0}^{n} \frac{(\nu - k - 1)!}{k!}\left(\frac{-t^2}{4}\right)^k + (-1)^{\nu+1} \ln(t/2) I_\nu(t)$$

$$- \tfrac{1}{2}(-t/2)^\nu \sum_{k=0}^{\infty} [\psi(k+1) + \psi(\nu + k + 1)] \frac{(t/2)^{2k}}{k!(\nu + k)!} \tag{8}$$

where $\psi(z)$ is the logarithmic derivative of the gamma function $\Gamma(z)$ and

$$I_\nu(t)=(t/2)^\nu\sum_{k=0}^{\infty}\frac{(t/2)^{2k}}{k!\Gamma(\nu+k+1)}. \tag{9}$$

We mention that the familiar non-relativistic and ultra-relativistic partition functions are regained from equation (5) if we substitute K_2 by the first terms of equations (7) and (8), respectively. Corrections to these limits are obtained if we retain higher-order terms in the asymptotic and ascending series for K_2.

The integral resulting from the substitution of equations (5)–(6) into equation (1) can be evaluated by closing the straight line contour to the right with a semi-arc of a circle. Because of the analytic behaviour of the K_ν, the integrand of equation (1) is an analytic function on the right-half t plane except at $t=n$; $n=1,2,3,\ldots$, where it has simple poles due to the factor $\cot(\pi t)$. The application of Cauchy's theorem gives the following expression for $\ln \Xi$:

$$\ln \Xi=\ln \Xi_b+\ln \Xi_s=\frac{m^2a^3}{\beta}\sum_{n=1}^{\infty}\frac{e^{\beta\xi n}}{n^2}K_2(n\beta m)+\sum_{n=1}^{\infty}\frac{e^{\beta\xi n}}{n}Z_s(n\beta). \tag{10}$$

From the requirement that the integral along this semi-circle vanishes when the radius goes to infinity one derives that $\xi\leqslant m$. This is an exact equation and it allows a rigorous discussion of the thermodynamic behaviour of the system. Other quantities of interest, such as the average number of particles present in the system, are obtained from (10) by suitable partial differentiation, e.g.

$$N=\frac{1}{\beta}\frac{\partial \ln \Xi}{\partial \xi}=\frac{m^2a^3}{\beta}\sum_{n=1}^{\infty}\frac{e^{\beta\xi n}}{n}K_2(n\beta m)+\sum_{n=1}^{\infty}e^{\beta\xi n}Z_s(n\beta). \tag{11}$$

3. The non-relativistic limit

The non-relativistic approximation ($\beta m \gg 1$) of equation (10) is obtained in a straightforward way by using the asymptotic expansion for the K_n to find

$$\ln \Xi=(\pi/2)^{1/2}\left(\frac{ma^2}{\beta}\right)^{3/2}\sum_{n=1}^{\infty}\frac{e^{-\beta n(m-\xi)}}{n^{5/2}}\left\{1+\frac{15}{8}(n\beta m)^{-1}+\frac{105}{128}(n\beta m)^{-2}+O([n\beta m]^{-3})\right\}$$
$$+4\sum_{p=1}^{\infty}\exp(-2\pi^2p^2ma^2/\beta)[\tfrac{1}{2}-(2\pi^2p^2ma^2/n\beta)] \tag{12}$$

where we have retained higher-order terms of equation (7) only in the bulk contribution. Using Dirichlet's series expansion for the Bose–Einstein (BE) functions:

$$F_\sigma(\alpha)=\sum_{n=1}^{\infty}n^{-\sigma}\exp(-n\alpha) \qquad \alpha\geqslant 0 \tag{13}$$

both bulk and surface contributions to $\ln \Xi$ can be rewritten as follows:

$$\ln \Xi=\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{ma^2}{\beta}\right)^{3/2}\bigg[F_{5/2}(\zeta)+\frac{15}{8}(\beta m)^{-1}F_{7/2}(\zeta)$$
$$+\frac{105}{128}(\beta m)^{-2}F_{9/2}(\zeta)+\cdots+\frac{1}{8\pi^4}\left(\frac{\lambda}{a}\right)^3\alpha^2F_1(\alpha)\bigg]. \tag{14}$$

The expression for the surface correction terms was further simplified because, as noted by Al'taie, it gives an important contribution only in the region around the transition temperature where $\zeta \simeq m$ so that the summation over n can be changed into an integration, where $\zeta = \beta(m-\xi)$, $\alpha = (16\pi^3\zeta)^{1/2}(a/\lambda)$ and $\lambda = (2\pi\beta/m)^{1/2}$ is the de Broglie thermal wavelength. We have kept only the leading term in 'a' in the finite size contribution to $\ln \Xi$.

In order to estimate the finite size correction and compare it with the relativistic corrections to the bulk term, we recall some properties of the BE functions.

The $F_\sigma(\alpha)$ are monotonically decreasing functions of α, and for $\sigma > 1$ they are finite at the origin. In particular, F_1 has a closed form

$$F_1(\alpha) = -\ln(1-\mathrm{e}^{-\alpha}). \tag{15}$$

For larger values of $\alpha(\alpha \geqslant 2)$ they rapidly merge into the exponential function, $F_\sigma(\alpha) \sim \exp(-\alpha)$. So in the neighbourhood of the origin $\alpha^2 F_1(\alpha)$ raises from zero as $\alpha^2 \ln \alpha$ and goes to zero as $\alpha^2 \exp(-\alpha)$ when $\alpha \to \infty$ with a finite maximum at $\alpha = \alpha_0$, where α_0 is the solution of the equation

$$2F_1(\alpha_0) - \alpha_0 F_0(\alpha_0) = 0 \tag{16}$$

and

$$F_0(\alpha) = [\exp(\alpha) - 1]^{-1}. \tag{17}$$

An upper bound for α_0 is obtained making use of the fact that $F_{\sigma'}(\alpha) > F_\sigma(\alpha)$ for $\sigma' < \sigma$ and $\alpha \in (0, \infty)$. Therefore,

$$2/\alpha_0 = F_0(\alpha_0)/F_1(\alpha_0) > 1. \tag{18}$$

An upper bound for the finite size contribution follows immediately since $\alpha_0^2 F_1(\alpha_0) < \alpha_0^2 F_0(\alpha_0) < \alpha_0$. Hence, from equation (18),

$$|\ln \Xi_\mathrm{s}| = \frac{\alpha^2 F_1(\alpha)}{4\pi^2} < \frac{1}{2\pi^2}. \tag{19}$$

The first relativistic correction can be rewritten as

$$\ln \Xi_\mathrm{R} \cong \frac{15}{8} N_0 \left(\frac{kT}{m}\right) F_{7/2}(\zeta)/F_{3/2}(\zeta) \tag{20}$$

where N_0 is the leading term in the expression for the number of particles:

$$N = \left(\frac{\pi}{2}\right)^{1/2} \left(\frac{ma^2}{\beta}\right)^{3/2} \left[F_{3/2}(\zeta) + \frac{15}{8}\left(\frac{kT}{m}\right) F_{5/2}(\zeta) + \cdots - (4\pi\zeta)^{1/2} F_0(\alpha)\right]. \tag{21}$$

The finite size contribution will be comparable with the first relativistic correction if $N_0(\beta m)^{-1} \sim 1$. This condition will be verified only in very rarified gases even considering heavy particles and low temperatures. For $m \sim 10^{-24}$ g and $T \sim 1$ K, N_0 has to be of order of 10^{13}. Here we have approximated the ratio of the BE functions in equation (20) by unity. To examine the effect of the relativistic correction on the critical temperature we set the chemical potential equal to the ground-state energy in equation (21):

$$N = \left(\frac{\pi}{2}\right)^{1/2} (ma^2 kT_\mathrm{c})^{3/2} \left[F_{3/2}(0) + \frac{15}{8}\left(\frac{kT_\mathrm{c}}{m}\right) F_{5/2}(0)\right]. \tag{22}$$

The non-relativistic critical temperature $\tilde{T}_c$ is given by

$$N_0=\left(\frac{\pi}{2}\right)^{1/2}(ma^2k\tilde{T}_c)^{3/2}F_{3/2}(0). \tag{23}$$

Hence, to first order we get

$$\tilde{T}_c^{3/2}=T_c^{3/2}\left[1+\frac{15}{8}\left(\frac{kT_c}{m}\right)F_{5/2}(0)/F_{3/2}(0)\right] \tag{24}$$

and since the term in the bracket is always greater than one the relativistic correction has the effect of decreasing the transition temperature, i.e. in the oppposite direction of the change caused by the finite size term, as verified by Al'taie.

4. The ultra-relativistic limit

An expression for $\ln \Xi$ in this limit ($\beta m \ll 1$) is obtained using equations (8) and (9) for the modified Bessel function in the general equation (10). The resulting expression can once again be written in terms of the BE functions:

$$\begin{aligned}\ln \Xi = 2(a/\beta)^3\Big\{&F_4(\zeta)-\tfrac{1}{4}(\beta m)^2F_2(\zeta)+\tfrac{1}{16}(\beta m)^4\\ &\times\left[F_0(\zeta)(A-\ln(m\beta))-\sum_{p=1}^{\infty}\exp(-p\zeta)\ln p\right]+\mathrm{O}([m\beta]^6)\\ &+F_0(\zeta)\frac{G(ma)}{(ma)^2}(\beta m)^4\Big\}\end{aligned} \tag{25}$$

where $A=[\psi(1)+\psi(3)]/2$, $\zeta=-\beta\xi$. We have defined $G(ma)/(ma)^2$ in the finite size contribution to be

$$\frac{G(ma)}{(ma)^2}=\sum_{p=1}^{\infty}\left[\frac{K_2(2\pi pma)}{(2\pi pma)^2}-\frac{K_3(2\pi pma)}{(2\pi pma)}\right] \tag{26}$$

which is obtained approximating the argument of the Bessel's functions in equation (6) by $\tau\simeq 2\pi pma$. For $ma\equiv 2\pi a/\lambda_{\text{Compton}}>1$, equation (26) can be further simplified to give

$$\frac{G(ma)}{(ma)^2}\simeq-\left(\frac{2}{\pi}\right)^{1/2}\frac{e^{-2\pi ma}}{(2\pi ma)^{3/2}}. \tag{27}$$

The absolute value of this ratio is at best of order of unity. Therefore, the finite size correction once again is negligible compared with the first relativistic correction to the bulk term.

Since the BE functions are defined only for non-negative argument, we have to restrict the chemical potential to non-positive values which is compatible with the previous upper bound on ξ. For massive particles equation (25) describes, because of restrictions of ξ, essentially the high-temperature regime of the gas, which is the only regime we can treat analytically.

5. Discussion

We have shown that taking into account relativistic corrections to the bulk contribution of $\ln \Xi$ is more important than the effects coming from the finite size of the system. This feature manifests itself also in the ultra-relativistic limit. In both cases, the finite size corrections will compete only with higher-order corrections to the leading bulk term. We feel that because of its importance the ultra relativistic limit deserves a separate presentation which shall be given in the future. (Aragão de Carvalho and Goulart Rosa)

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